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A Stochastic Model of Maxwell's Equations in 1 + 1 Dimensions

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We show that the random walk model due to Mark Kac which underlies the telegraph equations may be modified to produce Maxwell's field equations in 1 + 1 dimensions. This provides the field equations with a representation in terms of classical particles. It also establishes the Kac model as a strong conceptual link between the diffusion, telegraph, and Maxwell equations, and suggests that recent simulations of the Schrödinger and Dirac equations are analogous to Maxwell's equation in terms of interpretation.

There is now some evidence to suggest that the single-particle-continuous-trajectory paradigm may not be out of the question for quantum mechanics (Ord, 1992, 1993a; McKeon and Ord, 1992). In particular, there have been some recent results showing that the Schrödinger and Dirac equations are approximate descriptions of second-order effects in the trajectories of classical Brownian particles (Ord, 1996 a, b, c). In *these contexts* "wave functions" are observable properties of *ensembles* of Brownian particles, and the models themselves provide "classical" many-particle simulations of the equations of quantum mechanics based on a particle (as opposed to wave) picture.

Given these results it seems reasonable to ask the question, "Do Maxwell's equations have an underlying stochastic model based on ensembles of particles?"

The answer to this question is yes! The stochastic model is a simple modification of a model due to Kac (1974), which he showed provided a basis for the telegraph equations.

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We start with a space-time lattice in 1 + 1 dimensions with lattice spacing Δx and Δt. Assume all particles move either to the left or right with speed c and there is a source of particles specified by $a(x, t)$.

If $F^{\pm}(x, t)$ represents particle density on the lattice, then the Kac model gives

$$F^+(x, t + \Delta t) = F^+(x - \Delta x, t) + a(x, t)\Delta t$$
$$F^-(x, t + \Delta t) = F^-(x + \Delta x, t) + a(x, t)\Delta t \tag{1}$$

where $a(x, t)$ is a source term for particles. Equations (1) just express the fact that each particle propagates at constant speed in a unique direction, and that particles are emitted in opposite directions with equal frequency. To lowest order in Δx and Δt these equations give

$$\frac{\partial F^+}{\partial t}\Delta t = -\frac{\partial F^+}{\partial x}\Delta x + a(x, t)\Delta t$$
$$\frac{\partial F^-}{\partial t}\Delta t = \frac{\partial F^-}{\partial x}\Delta x + a(x, t)\Delta t \tag{2}$$

Writing

$$A(x, t) = \frac{1}{2}(F^+(x, t) + F^-(x, t))$$
$$\phi(x, t) = \frac{1}{2}(F^+(x, t) - F^-(x, t)) \tag{3}$$

we find that in terms of A and ϕ, (2) implies

$$\frac{\partial A}{\partial t} = -c\frac{\partial \phi}{\partial x} + a(x, t) \tag{4}$$

$$\frac{\partial \phi}{\partial t} = -c\frac{\partial A}{\partial x} \tag{5}$$

where $\Delta x/\Delta t = c$.

To decouple equation (4), differentiate the first with respect to t and the second with respect to x to give

$$\frac{\partial^2 A}{\partial t^2} = c^2\frac{\partial^2 A}{\partial x^2} + \frac{\partial a}{\partial t} \tag{6}$$

Similarly, eliminating A gives

$$\frac{\partial^2 \phi}{\partial t^2} = c^2\frac{\partial^2 \phi}{\partial x^2} - c\frac{\partial a}{\partial x} \tag{7}$$

Equations (5)–(7) are equivalent to Maxwell's field equations in 1 + 1 dimensions (Jackson, 1975). Here equation (5) is the Lorentz condition:

$$\frac{\partial A}{\partial x} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \tag{8}$$

Writing

$$\frac{1}{c}\frac{\partial a}{\partial t} = 4\pi J \tag{9}$$

we find that (6) becomes the wave equation for the 'vector' potential A

$$\frac{\partial^2 A}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi J}{c} \tag{10}$$

and similarly writing

$$\frac{1}{c}\frac{\partial a}{\partial x} = -4\pi\rho \tag{11}$$

we find that (7) becomes the wave equation for the scalar potential ϕ

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho \tag{12}$$

The two definitions (9) and (11) imply that

$$\frac{\partial J}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \tag{13}$$

which is the continuity equation.

Notice here that although we have a form of Maxwell's field equations in 1 + 1 dimensions, we obtained them *only* through the use of counting arguments. Equations (1) from which (9), (10), and (12) follow themselves arise from counting particles on a lattice (Kac, 1974).

Generalizing (9), (10), and (12) to 3 + 1 dimensions is easy to accomplish formally. Writing A and J as three component vectors and replacing $\partial/\partial x$ and $\partial^2/\partial x^2$ by ∇ and ∇^2, respectively, does the trick. However, it remains to justify this in terms of the stochastic model.

Since we generally think of Maxwell's field equations as a system of equations describing waves, we can ask at this point, "Where do the wave aspects come from in this description of particles?" The answer is that the quantities A and ϕ above are ensemble averages and they must derive their wave aspects from the ensemble. Each separate particle propagates simply as $x(t) = x_0 \pm ct$. The concepts of "wavelength" and "frequency" come about from the Fourier decomposition of the initial conditions of the particle densities.

Unlike "real photons," our particles have no analog of frequency, wavelength, or interference properties by themselves, and only ensemble averages can be usefully described by the "field" equations. This is very similar to the recent results on simulations of quantum mechanics (Ord, 1996a, b, c), where the wave functions came out as descriptions of expectations of correlations over ensembles of classical particles. Interference effects arose as a result of ensemble averages, and not (as in quantum mechanics) as a result of properties of the "particles" themselves.

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